

TENSOR FIELDS ON SELF-DUAL WARPED AdS_3 BACKGROUND

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Abstract

Considering rank s fields obey first order equation of motion, we study the dynamics of such fields in a 3 dimensional self-dual space-like warped AdS_3 black hole background. We argue that in this background, symmetric conditions and gauge constraint can not be satisfied simultaneously. Using new suitable constraint, we find the exact solutions of equation of motion. Then, we obtain the quasi-normal modes by imposing appropriate boundary condition at horizon and infinity.

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1 Introduction

Quantum Gravity as a theory of consistently quantized symmetric massless rank 2 field is one of the most challenging subjects for high energy physicist during the last decade. An outstanding efforts have been done for overcoming this problem and some elegant and interesting theories such as supersymmetric field theory, string theory, etc, invented and many insights gained but the problem is unsolved completely up to now.

The recently attracted subject in this field was studying the gravity in three dimensions. Although, it seems that Einstein gravity is trivial in three dimensions[1] but, by adding higher correction to usual Einstein action with cosmological constant, one obtains theories which have propagating degrees of freedom[2].

One of such theories is Topologically Massive Theory(TMG) which its higher derivative terms are gravitational Chern-Simons term[3, 4]

$$I_{TMG} = \frac{1}{16\pi G} \left(I_{EH} + \frac{1}{\mu} I_{CS} \right) \quad (1)$$

with μ is coupling constant and I_{EH} includes the cosmological term $-2\Lambda = \frac{2}{l^2}$.

TMG admits maximally symmetric solutions AdS_3 with $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry[5] and BTZ black hole[6] and also solutions with fewer symmetries known as warped AdS_3 and its corresponding black holes[7]. Due to hidden conformal symmetries of the propagating modes on the warped background, it was conjectured that such gravitational warped solution has a dual conformal field theory with the corresponding central charges[5]. Motivated by this observations, many works has been done in this line for better understanding the duality and other related topics of quantum gravity and black holes in three dimensions[8].

Self-dual warped AdS_3 solution is also solution of TMG[9]. It may be defined as real line fibrations over AdS_2 which preserves a single $SL(2, \mathbb{R})$ isometry and a non-compact $U(1)$ isometry generated by translation along the fibre coordinate.

Such background has some very interesting properties and its several aspects such as geometrical properties, thermodynamics, its CFT dual and etc, has been studied. Especially, using an algebraic way, the author of [10] were found the quasi-normal modes of scalar, vector and tensor perturbations for AdS_3 , BTZ and warped AdS_3 background. This calculation has been done using other methodes such as finding the exact solution[11]. For example, in

[12, 13], the authors were able to find the exact solution in BTZ black hole background for any general integer or half-integer rank s field and found the quasi-normal modes.

In fact, till recently, because of some no-go theorems [14], efforts typically was focused on the dynamics of fields with spin lower than 2 in gravitational backgrounds.

But, due to the work of [15] the Higher Spin fields were entered to the game and seems that have somehow serious rolls in fundamental physics. Various aspects of theories with higher spin fields such as finding a Lagrangian formalism, studying the conserved current and so on have been studied, see for example[16].

Our aim in this paper is to continue this research line by studying with details the constraint in which a rank s field should satisfy in this background and obtaining the exact solutions of tensor mode perturbations and finally, finding the quasi-normal modes. We will also study modes with spin greater than 2(higher spin fields) on this background.

For self-dual warped AdS_3 , we suppose that the main equation in which higher spin fields should obey is

$$\epsilon_\mu^{\alpha\beta} \nabla_\alpha \Phi_{\beta\nu_2\nu_3\cdots\nu_s} = -m\Phi_{\mu\nu_2\nu_3\cdots\nu_s}. \quad (2)$$

Due to lower symmetry in this background, so is not locally AdS and it was shown in [17] that such a simple first order equation can not be carried out for a metric perturbation but, one can still consider the above equation for a massive spin s field in the given background[10].

Nevertheless, in spite of BTZ background[13], such a linear equation does not imply the following simple form of second order Klein-Gordon equation and gauge condition for all components of a spin s field² [18, 19]

$$(\nabla^2 - m_s^2)\Phi_{\mu_1\mu_2\mu_3\cdots\mu_s} = 0, \quad (3)$$

$$\nabla^\mu \Phi_{\mu\mu_2\cdots\mu_s} = 0. \quad (4)$$

In fact the situation is worse noting that the linear equation and the gauge constraint for a symmetric field satisfied only for zero mode(s-wave).

However, we will show that by imposing a suitable weaker condition, one can write the equation for some components in the Klein-Gordon form and can solve them exactly. In particular, for $\Phi_{\theta\theta\cdots\theta}$ and $\Phi_{t\theta\cdots\theta}$ components, the

²Note that the symmetric rank s field obey traceless condition as $g^{\mu\nu}\Phi_{\mu\nu\mu_3\cdots\mu_s} = 0$

above equations can be modified slightly and solve it exactly. Then using the linear equation(2) and some contiguous relations between the hypergeometric functions, one can obtain the solution for the other components and quasi normal modes and also the left and right conformal weight.

The paper organized as follows. In section 2, we briefly introduce the three dimensional self dual Warped AdS_3 background. In the next section, we discuss about the equation of motion and gauge constraint of higher spin fields in this background and will see that all components do not satisfy Klein-Gordon equation and harmonic constraint. In section four, we find the exact solutions for spin 2 case for two components $h_{\theta\theta}$ and $h_{t\theta}$ and then using the first order equation (2) we find the solution for $h_{\theta t}$ and h_{tt} . After that, by imposing Dirichlet Boundary condition, we find the quasi-normal modes for massive spin 2 field. In section 5 we generalize our computations for higher spin field and find the quasi normal modes.

2 Self Dual Warped AdS_3 Background

In this section, we briefly introduce self-dual warped AdS3 solution. *Self dual warped AdS_3* is a solution of equation of motion of Topological Massive Gravity(TMg) in three dimensions[9]. The metric is given by

$$ds^2 = \frac{l^2}{\nu^2 + 3} \left(- (x - x_+) (x - x_-) d\tau^2 + \frac{1}{(x - x_+) (x - x_-)} dx^2 + \frac{4\nu^2}{\nu^2 + 3} \left(\alpha d\theta + \frac{1}{2} (2x - x_+ - x_-) d\tau \right)^2 \right), \quad (5)$$

where x_+ and x_- are the outer and inner horizons radius respectively.

In fact, the warped vacua of TMG classified into three spacelike, timelike and null warped types. Classification depends on the whether the norm of the Killing vector generating the $U(1)$ isometry is positive, negative or zero. First two types can also be classified as stretched ($\mu l > 3$) or squashed ($\mu l < 3$) depending on the magnitude of the warp factor. These background spacetimes are not locally AdS_3 .

Quotients of warped AdS3 along various Killing directions may give rise to black holes[5]. Black hole solutions free of closed timelike curves (CTCs) can only be found in spacelike stretched and null warped AdS_3 . Self-dual solutions in AdS_3 is quotients of spacelike warped AdS_3 along the $U(1)$. Such geometries have Killing horizons and no CTCs[20] .

It is shown in [9] that under the consistent boundary condition, the $U(1)$ isometry is enhanced to a Virasoro algebra with nonvanishing left central charge while the $SL(2, \mathbb{R})$ isometry becomes trivial with the vanishing right central charge,

$$c_L = \frac{4\nu l}{\nu^2 + 3}, \quad c_R = 0. \quad (6)$$

It is conjectured that the self-dual warped AdS_3 black hole is dual to a two dimensional chiral CFT, which provides an example of warped AdS/CFT dual.

The left and right temperatures of CFT are defined by [9, 21]

$$T_L = \frac{\alpha}{2\pi l}, \quad T_R = \frac{x_+ - x_-}{4\pi l} \quad (7)$$

The mass M and angular momentum J of this black hole are given by

$$M = 0, \quad J = \frac{(\alpha^2 - 1)\nu}{6G(\nu^2 + 3)}. \quad (8)$$

The angular velocity of the event horizon Ω_H and the Bekenstein-Hawking entropy S_{BH} of this solution are respectively given by

$$\begin{aligned} \Omega_H &= -\frac{x_+ - x_-}{2\alpha}, \\ S_{BH} &= \frac{2\pi\alpha\nu}{3G(\nu^2 + 3)}. \end{aligned} \quad (9)$$

Note also that, this solution is asymptotic to the spacelike warped AdS_3 spacetime and under a suitable coordinate transformation the metric of self-dual warped AdS_3 black hole can be transformed to the metric of spacelike warped AdS_3 spacetime

$$ds^2 = \frac{1}{\nu^2 + 3} \left(-\cosh^2 \sigma dv^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} (du + \sinh \sigma dv)^2 \right). \quad (10)$$

At the end, let us mention that since the self-dual warped AdS_3 is not locally AdS its curvature, Riemann and Ricci tensor can not be written in the usual form in terms of metric³. In fact, one can see that for background (5)

$$R_{\mu\nu\lambda\theta} = R_{\mu\nu\lambda\theta}^0 + r_{\mu\nu\lambda\theta}, \quad (11)$$

³In a maximally symmetric space one has $R_{\mu\nu\lambda\theta} = g_{\mu\theta}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\theta} = \epsilon_{\mu\nu\alpha}\epsilon_{\lambda\theta\beta}G^{\alpha\beta}$ where $G^{\alpha\beta}$ is Einstein tensor and we have defined $\epsilon^{xt\theta} = +\frac{1}{\sqrt{-g}}$.

where

$$\begin{aligned} R_{\mu\nu\lambda\theta}^0 &= \frac{\lambda^2}{4}(g_{\mu\theta}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\theta}), \\ r_{rtrt} &= 1 - \lambda^2, \quad r_{other} = 0 \end{aligned} \quad (12)$$

and we have defined $\lambda^2 = \frac{4\nu^2}{\nu^2+3}$. For later use, we also define $x_0 = \frac{x_++x_-}{2}$ and $\bar{x}_0 = \frac{x_+-x_-}{2}$.

Moreover, there are some relations between metric components and Christoffel symbols as⁴

$$\begin{aligned} g^{xx} \Gamma_{x\theta}^\theta &= -g^{\theta t} \Gamma_{\theta t}^x, & g^{tt} \Gamma_{\theta t}^x &= -g^{xx} \Gamma_{x\theta}^t, \\ g^{xx} \Gamma_{xt}^\theta &= -g^{\theta\theta} \Gamma_{\theta t}^x - g^{\theta t} \Gamma_{tt}^x \\ g^{xx} \Gamma_{xt}^t &= -g^{\theta t} \Gamma_{\theta t}^x - g^{tt} \Gamma_{tt}^x. \end{aligned} \quad (13)$$

These relations are very useful and important in finding the final equations. In fact, our discussion about the equations and quasi normal modes can be generalized to any other metrics in 3 dimensions in which $g_{x\mu} = 0, \mu \neq x$ and satisfied (13).

Without loose of generality, we will set $\frac{l^2}{\nu^2+3} = 1$ and $\alpha = 1$.

3 Fields on Self-dual Warped AdS_3

Massive integer spin s fields in AdS_3 spaces are realized by totally symmetric tensors of rank s satisfying the following equation of motion, gauge condition and traceless condition as[18, 19]

$$(\nabla^2 - m_s^2)\Phi_{\mu_1\mu_2\mu_3\cdots\mu_s} = 0, \quad (14)$$

$$\nabla^\mu \Phi_{\mu\mu_2\cdots\mu_s} = 0, \quad (15)$$

$$g^{\mu\nu} \Phi_{\mu\nu\mu_3\cdots\mu_s} = 0. \quad (16)$$

Here, $\Phi_{\mu_1\mu_2\cdots\mu_s}$ is a totally symmetric rank s tensor and for AdS_3 the mass of the field is given by $m_s^2 = s(s-3) + M^2$. The first term is the mass that exists due to the curvature of AdS_3 .

The above set of equations in a maximally symmetric space are equivalent with the following first order equation [13]

$$\epsilon_\mu^{\alpha\beta} \nabla_\alpha \Phi_{\beta\nu_2\nu_3\cdots\nu_s} = -m\Phi_{\mu\nu_2\nu_3\cdots\nu_s}. \quad (17)$$

⁴These relations can be recasted as $g^{xx} \Gamma_{x\nu}^\mu = -g^{\mu\beta} \Gamma_{\beta\nu}^x$.

where $m^2 = M^2 + (s - 1)^2$.

However, it was shown[17] in self-dual warped AdS_3 space, where the background is not locally AdS , one can not rewrite the field equations for metric perturbations in a simple form as (2). But, one can yet consider any massive tensor field $\Phi_{\nu_1\nu_2\nu_3\cdots\nu_s}$ obeys the above first order equation[10]. Adopting (2) for a tensor field $\Phi_{\nu_1\nu_2\nu_3\cdots\nu_s}$, one may search a set of equations similar to (3). Unfortunately, because of the reasons which will be presented bellow, the gauge constraint and the tracelessness condition do not satisfied for a general "symmetric" rank s tensor field $\Phi_{\nu_1\nu_2\nu_3\cdots\nu_s}$. Indeed, the Klein-Gordon equation can not be written in the simple form (3) for all components of a given field.

The argument is as follows. Let us for simplicity consider a rank 2 field $h_{\mu\nu}$ and suppose that is totally symmetric and obeys the equation (2). It means that that

$$\begin{aligned} 0 &= \epsilon^{\mu\nu\rho} \epsilon_\mu^{\alpha\beta} \nabla_\alpha h_{\beta\nu} \\ &= -g^{\rho\beta} \nabla^\alpha h_{\beta\alpha} + g^{\rho\alpha} \nabla_\alpha g^{\nu\beta} h_{\beta\nu} \end{aligned} \quad (18)$$

In the other hand, using (2) for h_{rr} one obtains

$$\begin{aligned} -mh_{rr} &= g_{rr} \epsilon^{rt\theta} (\nabla_t h_{\theta r} - \nabla_\theta h_{tr}) \\ &= g_{rr} \epsilon^{rt\theta} (\nabla_t h_{r\theta} - \nabla_\theta h_{rt}) \\ &= g_{rr} \epsilon^{rt\theta} (\nabla_r (h_{t\theta} - h_{\theta t}) - mg_{rr} \epsilon^{rt\theta} (g_{tt} h_{\theta\theta} - g_{t\theta} h_{t\theta} - g_{\theta t} h_{\theta t} + g_{\theta\theta} h_{tt})) \\ &= \frac{m}{g^{rr}} (0 + g^{\theta\theta} h_{\theta\theta} + 2g^{t\theta} h_{t\theta} + g^{tt} h_{tt}) \end{aligned} \quad (19)$$

In the third line of (19) we have used (2) for some other components⁵ and the last line is true for the metric (5). So, for the self-dual warped AdS_3 metric a symmetric rank s field automatically satisfies the tracelessness condition i.e we have $g^{\mu\nu} \Phi_{\nu\mu\nu_3\cdots\nu_s} = 0$. Therefore, the second line in (18) is equal to zero provided that the gauge constraint be also equal to zero.

⁵Notice that (2) can be written as

$$\nabla_\lambda h_{\alpha\nu} - \nabla_\alpha h_{\lambda\nu} = m \epsilon_{\lambda\alpha}^{\mu} h_{\mu\nu}$$

But, for gauge constraint, one has

$$\begin{aligned}
-m \nabla^\mu \Phi_{\mu\nu_2\nu_3\cdots\nu_s} &= \epsilon^{\mu\alpha\beta} \nabla_\mu \nabla_\alpha \Phi_{\beta\nu_2\nu_3\cdots\nu_s} = \frac{1}{2} \epsilon^{\mu\alpha\beta} [\nabla_\mu, \nabla_\alpha] \Phi_{\beta\nu_2\nu_3\cdots\nu_s} \quad (20) \\
&= \epsilon^{\mu\alpha\beta} g^{\rho\rho'} R_{\beta\rho\mu\alpha}^0 \Phi_{\rho'\nu_2\nu_3\cdots\nu_s} + \epsilon^{\mu\alpha\beta} g^{\rho\rho'} R_{\nu_2\rho\mu\alpha}^0 \Phi_{\beta\rho'\cdots\nu_s} + \cdots \\
&+ \epsilon^{\mu\alpha\beta} g^{\rho\rho'} r_{\beta\rho\mu\alpha} \Phi_{\rho'\nu_2\nu_3\cdots\nu_s} + \epsilon^{\mu\alpha\beta} g^{\rho\rho'} r_{\nu_2\rho\mu\alpha} \Phi_{\beta\rho'\cdots\nu_s} + \cdots.
\end{aligned}$$

As it is obvious, the usual harmonic gauge constraint does not satisfied for all components of a general rank s field $\Phi_{\mu\nu_2\nu_3\cdots\nu_s}$. In particular, for a spin 2 field $h_{\mu\nu}$ one obtains

$$-m \nabla^\mu h_{\mu\theta} = 0 \quad (21)$$

$$-m \nabla^\mu h_{\mu t} = -2(1 - \lambda^2) g^{xx} h_{x\theta} \quad (22)$$

$$-m \nabla^\mu h_{\mu x} = 2(1 - \lambda^2) (g^{tt} h_{t\theta} + g^{t\theta} h_{\theta\theta}). \quad (23)$$

Thus, the symmetric condition on $h_{\mu\nu}$ at least implies that $h_{x\theta} = 0$. But, from (2) we have

$$h_{x\theta} = \frac{g_{xx}}{-m\sqrt{-g} + g_{xx}\Gamma_{t\theta}^x} (\partial_t h_{\theta\theta} - \partial_\theta h_{t\theta}) \quad (24)$$

If we consider the ansatz

$$h_{\mu\nu}(x, \theta, \phi) = e^{-i(\omega t - k\theta)} R_{\mu\nu}(x) \quad (25)$$

then $h_{x\theta} = 0$ means that $\omega = k = 0$ which is not interesting generally. So, we consider in the rest of the paper the field $h_{\mu\nu}$ is not totally symmetric but is traceless in the sense that

$$g^{rr} h_{rr} + g^{\theta\theta} h_{\theta\theta} + g^{t\theta} (h_{t\theta} + h_{\theta t}) + g^{tt} h_{tt} = 0 \quad (26)$$

This implies that we should impose following conditions

$$\nabla_i (h_{jr} - h_{rj}) = 0, \quad (27)$$

where $\{i, j\} = \{t, \theta\}$. To proceed we also consider one of the gauge constraint

$$\nabla^\mu h_{\mu\theta} = 0, \quad (28)$$

yet can be satisfied. This help us to simplify and write the equation of motion for $h_{\theta\theta}$ and $h_{t\theta}$ in a simple form. In fact, the three gauge constraint and one

tracelessness condition in symmetric case replaced to three conditions (27) and a gauge constraint (28). The above conditions can be generalized for a higher spin as following

$$\nabla_i(\Phi_{j\dots r\dots} - \Phi_{r\dots j\dots}) = 0, \quad (29)$$

$$g^{\alpha\beta}\Phi_{\nu_1\dots\alpha\dots\beta\dots\nu_s} = 0, \quad (30)$$

$$\nabla^\mu\Phi_{\mu\theta\dots\theta} = 0. \quad (31)$$

Let us look for a Klein-Gordon equation for fields. From (2) one finds

$$\begin{aligned} (\nabla^2 - m^2)\Phi_{\mu\nu_2\nu_3\dots\nu_s} &= \nabla^\sigma\nabla_\mu\Phi_{\sigma\nu_2\nu_3\dots\nu_s} \\ &= g^{\sigma\rho}[\nabla_\rho, \nabla_\mu]\Phi_{\sigma\nu_2\nu_3\dots\nu_s} + \nabla_\mu\nabla^\sigma\Phi_{\sigma\nu_2\nu_3\dots\nu_s} \\ &= g^{\sigma\rho}g^{\eta\delta}(R_{\sigma\delta\rho\mu}\Phi_{\eta\nu_2\nu_3\dots\nu_s} + R_{\nu_2\delta\rho\mu}\Phi_{\sigma\eta\nu_3\dots\nu_s} \\ &\quad + R_{\nu_3\delta\rho\mu}\Phi_{\sigma\nu_2\eta\nu_4\dots\nu_s} + \dots + R_{\nu_s\delta\rho\mu}\Phi_{\sigma\nu_2\nu_3\dots\nu_{s-1}\eta}) \\ &\quad + \nabla_\mu\nabla^\sigma\Phi_{\sigma\nu_2\nu_3\dots\nu_s} \\ &= -\frac{\lambda^2}{4}(s+1)\Phi_{\mu\nu_2\nu_3\dots\nu_s} \\ &\quad + g^{\sigma\rho}g^{\eta\delta}(r_{\sigma\delta\rho\mu}\Phi_{\eta\nu_2\nu_3\dots\nu_s} + r_{\nu_2\delta\rho\mu}\Phi_{\sigma\eta\nu_3\dots\nu_s} \\ &\quad + r_{\nu_3\delta\rho\mu}\Phi_{\sigma\nu_2\eta\nu_4\dots\nu_s} + \dots + r_{\nu_s\delta\rho\mu}\Phi_{\sigma\nu_2\nu_3\dots\nu_{s-1}\eta}) \\ &\quad + \nabla_\mu\nabla^\sigma\Phi_{\sigma\nu_2\nu_3\dots\nu_s}. \end{aligned} \quad (32)$$

Especially, for $\Phi_{\theta\theta\dots\theta}$ and $\Phi_{t\theta\dots\theta}$ we obtain

$$(\nabla^2 - m^2)\Phi_{\theta\theta\dots\theta} = -\frac{\lambda^2}{4}(s+1)\Phi_{\theta\theta\dots\theta}, \quad (33)$$

$$(\nabla^2 - m^2)\Phi_{t\theta\dots\theta} = -\frac{\lambda^2}{4}(s+1)\Phi_{t\theta\dots\theta} + (1 - \lambda^2)g^{xx}g^{t\eta}\Phi_{\eta\theta\dots\theta}. \quad (34)$$

4 Rank 2 Field in Self-dual Warped AdS3

In this section, we consider a rank 2 tensor field and study its dynamics in self dual warped $AdS3$ background. We solve the equations of motion and then find the quasi normal modes. At last, we will do the computation for the extremal case.

4.1 Field Dynamics

Here, we consider a traceless rank 2 field $h_{\mu\nu}$ which should satisfy (2). Here, for later use, we present this equation in details using the notation of [10] as

following

$$\begin{aligned}
\partial_x h_{\theta\theta} &= \partial_\theta h_{r\theta} + \Gamma_{(\theta\theta)} + m_{(\theta\theta)}, \\
\partial_x h_{t\theta} &= \partial_t h_{r\theta} + \Gamma_{(t\theta)} + m_{(t\theta)}, \\
\partial_x h_{\theta t} &= \partial_\theta h_{rt} + \Gamma_{(\theta t)} + m_{(\theta t)}, \\
\partial_x h_{tt} &= \partial_t h_{rt} + \Gamma_{(tt)} + m_{(tt)}, \\
-mh_{x\theta} &= g_{xx}\epsilon^{xt\theta}(\partial_t h_{\theta\theta} - \partial_\theta h_{t\theta}) + \Gamma_{(x\theta)}, \\
-mh_{xt} &= g_{xx}\epsilon^{xt\theta}(\partial_t h_{\theta t} - \partial_\theta h_{tt}) + \Gamma_{(xt)}, \\
-mh_{xx} &= g_{xx}\epsilon^{xt\theta}(\partial_t h_{\theta x} - \partial_\theta h_{tx}) + \Gamma_{(xx)},
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\Gamma_{(\theta\theta)} &= \Gamma_{r\theta}^\lambda h_{\theta\lambda} - \Gamma_{\theta\theta}^\lambda h_{r\lambda}, \\
\Gamma_{(t\theta)} &= \Gamma_{r\theta}^\lambda h_{t\lambda} - \Gamma_{t\theta}^\lambda h_{r\lambda}, \\
\Gamma_{(\theta t)} &= \Gamma_{rt}^\lambda h_{\theta\lambda} - \Gamma_{\theta t}^\lambda h_{r\lambda}, \\
\Gamma_{(tt)} &= \Gamma_{rt}^\lambda h_{\theta\lambda} - \Gamma_{tt}^\lambda h_{r\lambda}, \\
\Gamma_{(x\theta)} &= g_{xx}\epsilon^{xt\theta}(-\Gamma_{t\theta}^\lambda h_{\theta\lambda} + \Gamma_{\theta\theta}^\lambda h_{t\lambda}), \\
\Gamma_{(xt)} &= g_{xx}\epsilon^{xt\theta}(-\Gamma_{tt}^\lambda h_{\theta\lambda} + \Gamma_{\theta t}^\lambda h_{t\lambda}), \\
\Gamma_{(xx)} &= g_{xx}\epsilon^{xt\theta}(-\Gamma_{tx}^\lambda h_{\theta\lambda} + \Gamma_{\theta x}^\lambda h_{t\lambda}),
\end{aligned} \tag{36}$$

$$\begin{aligned}
m_{(\theta\theta)} &= -mg_{xx}\epsilon^{\theta rt}(g_{t\theta}h_{\theta\theta} - g_{\theta\theta}h_{t\theta}), \\
m_{(t\theta)} &= -mg_{xx}\epsilon^{tr\theta}(g_{t\theta}h_{\theta t} - g_{tt}h_{\theta\theta}), \\
m_{(\theta t)} &= -mg_{xx}\epsilon^{\theta rt}(g_{t\theta}h_{\theta t} - g_{\theta\theta}h_{tt}), \\
m_{(tt)} &= -mg_{xx}\epsilon^{tr\theta}(g_{t\theta}h_{tt} - g_{tt}h_{\theta t}).
\end{aligned} \tag{37}$$

Next, we focus on the equations of $h_{\theta\theta}$ and $h_{t\theta}$ components which decouple from the other components and can be written as (33). Although, for finding the quasi-normal modes we should also find the exact solution for h_{tt} but, using (2), we will be able to find exact solutions for $h_{\theta t}$, h_{tt} , $h_{x\theta}$, h_{xt} and h_{xx} .

Recalling the fact that $\Gamma_{\theta\theta}^\lambda = 0$ and the ansatz (25) and using the constraint (29), one can find the exact solution as following. Firstly, one should solve (33) to find $h_{\theta\theta}$ and $h_{t\theta}$. Then, from the first two equations of (35) one can find $h_{\theta t}$ and $h_{r\theta}$. After that, from the last three equations of (35) one finds h_{tr} , $h_{r\theta}$ and especially h_{rt} in terms of $h_{\theta\theta}$, $h_{t\theta}$, $h_{\theta t}$ and h_{tt} . Finally, inserting the h_{rt} in the third equation of (35) one can obtain h_{tt} . In the following we present the results of the computations.

4.2 Solution for $h_{\theta\theta}$ and $h_{t\theta}$

The equations of motion for $h_{\theta\theta}$ and $h_{t\theta}$ reads as

$$(\nabla^2 - m^2)\Phi_{\theta\theta} = -3\frac{\lambda^2}{4}\Phi_{\theta\theta}, \quad (38)$$

$$(\nabla^2 - m^2)\Phi_{t\theta} = \left(\frac{\lambda^2}{4} - 1\right)\Phi_{t\theta} + (1 - \lambda^2)\left(x - \frac{x_+ + x_-}{2}\right)h_{\theta\theta}. \quad (39)$$

So, first of all, we should evaluate ∇^2 . As [13], one has

$$\begin{aligned} \nabla^2 h_{\mu\nu} &= \Delta h_{\mu\nu} \\ &- \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\Gamma_{\beta\mu}^\sigma)h_{\sigma\nu} - \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\Gamma_{\beta\nu}^\sigma)h_{\mu\sigma} \\ &- 2\Gamma_{\alpha\mu}^\rho g^{\alpha\beta}\nabla_\beta h_{\rho\nu} - 2\Gamma_{\alpha\nu}^\rho g^{\alpha\beta}\nabla_\beta h_{\mu\rho} \\ &- g^{\alpha\beta}\Gamma_{\beta\mu}^\sigma\Gamma_{\alpha\sigma}^\rho h_{\rho\nu} - g^{\alpha\beta}\Gamma_{\beta\nu}^\sigma\Gamma_{\alpha\sigma}^\rho h_{\mu\rho} \\ &- g^{\alpha\beta}\Gamma_{\alpha\mu}^\rho\Gamma_{\beta\nu}^\sigma h_{\sigma\rho} - g^{\alpha\beta}\Gamma_{\alpha\mu}^\rho\Gamma_{\beta\nu}^\sigma h_{\rho\sigma} \end{aligned} \quad (40)$$

where Δ is the usual scalar Laplacian

$$\Delta h_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta h_{\mu\nu}). \quad (41)$$

Calculations of the first and second line of (40) are straightforward. For the third line, one can see in a background where its metric are only x-dependent and $g_{xi} = 0$, only $\nabla_i h_{xj}$ and h_{xx} contribute to $\nabla^2 h_{ij}(\{i, j\} = t, \theta)$. Using the first order equation (2) and the constraint (27), one can rewrite the third line in terms of $h_{\theta\theta}$, $h_{t\theta}$, h_{tt} . For example

$$\nabla_t h_{xt} = m\frac{\sqrt{-g}}{-\hat{g}}(-g_{t\theta}h_{tt} + g_{tt}h_{t\theta}) + \nabla_x h_{tt}, \quad (42)$$

where $\hat{g} = g_{tt}g_{\theta\theta} - (g_{t\theta})^2$.

Moreover, by using (42) one can show that the first and second terms in the forth line of (40) are equal to zero and just the last expression would contribute to ∇^2 . In this expression we have h_{xx} term which can be replaced using traceless condition. Using the above equations and inserting the ansatz (25) one can obtain the following set of differential equations for $h_{\theta\theta}$ and $h_{t\theta}$

$$(\Delta \mathbf{1}_{2 \times 2} + M_{2 \times 2}) \begin{pmatrix} R_{t\theta} \\ R_{\theta\theta} \end{pmatrix} = 0, \quad (43)$$

where the mass matrix M is given by

$$M_{2 \times 2} = \begin{pmatrix} -m^2 + \frac{\lambda^2}{4} & (1 - \frac{2m}{\lambda})(1 - \lambda^2)(x - x_0) \\ 0 & -(m - \lambda)^2 + \frac{\lambda^2}{4} \end{pmatrix} \quad (44)$$

and the operator Δ is equal to

$$\Delta = (x - x_+)(x - x_-) \frac{\partial^2}{\partial x^2} + 2(x - x_0) \frac{\partial}{\partial x} + Q(x), \quad (45)$$

where

$$Q(x) = \frac{(\omega + k(x - x_0))^2}{(x - x_+)(x - x_-)} - \frac{k^2}{\lambda^2} \quad (46)$$

As it is clear, except for the $h_{\theta\theta}$, the equations (49) are a set of nonlinear coupled equations. For solving such coupled equations, one may try to diagonalize M and decouple the equations but, the components of M are x -dependent and we should be careful for doing such procedure. Overcoming such problem is easy by defining the following new fields

$$H_{t\theta} = \frac{R_{t\theta}}{(x - x_0)} \quad (47)$$

$$H_{\theta\theta} = R_{\theta\theta}. \quad (48)$$

Then, one can obtain the new set of equations as

$$(\delta_{2 \times 2} + \mathcal{M}_{2 \times 2}) \begin{pmatrix} H_{t\theta} \\ H_{\theta\theta} \end{pmatrix} = 0, \quad (49)$$

where the new matrix \mathcal{M} is a constant matrix and is given by

$$\mathcal{M}_{ij} = \frac{M_{ij}}{(x - x_0)^{j-i}}.$$

Also, the new diagonal differential operator δ is as

$$\delta_{11} = \Delta + 2 \frac{(x - x_+)(x - x_-)}{(x - x_0)} \frac{\partial}{\partial x} + 2, \quad (50)$$

$$\delta_{22} = \Delta, \quad (51)$$

and all other components of δ are zero. Now, we can diagonalize \mathcal{M} and find new decoupled equations. Noting that the eigenvalues of \mathcal{M} are \mathcal{M}_{11}

and \mathcal{M}_{22} , one can find the eigenvector and matrix transformation U such that $U\mathcal{M}U^{-1}$ becomes diagonal. The new fields in which we have decoupled differential equations are $\mathcal{H} = UH$ where

$$\mathcal{H}_{t\theta} = H_{t\theta} - \frac{1 - \lambda^2}{\lambda^2} H_{\theta\theta} \quad (52)$$

$$\mathcal{H}_{\theta\theta} = H_{\theta\theta}. \quad (53)$$

Doing the above prescription and using the following common change of variable

$$z = \frac{x - x_+}{x - x_-} \quad (54)$$

one finally obtains

$$\begin{aligned} z(1-z) \frac{\partial^2}{\partial z^2} \mathcal{H}_{t\theta} &+ \left(1 - z + \frac{4z}{1+z}\right) \frac{\partial}{\partial z} \mathcal{H}_{t\theta} \\ &+ \left(Q(z) + \frac{2}{1-z} + \frac{\mathcal{M}_{22}}{1-z}\right) \mathcal{H}_{t\theta} = 0, \end{aligned} \quad (55)$$

$$\begin{aligned} z(1-z) \frac{\partial^2}{\partial z^2} \mathcal{H}_{\theta\theta} &+ (1-z) \frac{\partial}{\partial z} \mathcal{H}_{\theta\theta} \\ &+ \left(Q(z) + \frac{\mathcal{M}_{33}}{1-z}\right) \mathcal{H}_{\theta\theta} = 0, \end{aligned} \quad (56)$$

where

$$Q(z) = -\frac{(\omega - \frac{x_+ - x_-}{2}k)^2}{(x_+ - x_-)^2} + \frac{(\omega + \frac{x_+ - x_-}{2}k)^2}{(x_+ - x_-)^2} \frac{1}{z} + \frac{k^2(1 - \frac{1}{\lambda^2})}{1-z}.$$

The solutions of the above equations can be written in terms of hypergeometric function as following

$$\begin{aligned} \mathcal{H}_{\theta\theta}(z) &= z^\alpha (1-z)^{\beta_{\theta\theta}+1} (C_{\theta\theta} F(a_{\theta\theta} + 1, b_{\theta\theta+1}, c; z)) \\ &\quad + D_{\theta\theta} F(a_{\theta\theta}^* + 1, b_{\theta\theta}^* + 1, c^*; z), \end{aligned} \quad (57)$$

$$\begin{aligned} \mathcal{H}_{t\theta}(z) &= z^\alpha \left(\frac{1-z}{1+z}\right) (1-z)^{\beta_{t\theta}+1} (C_{t\theta} F(a_{t\theta} + 1, b_{t\theta} + 1, c; z) \\ &\quad + D_{t\theta} F(a_{t\theta}^* + 1, b_{t\theta}^* + 1, c^*; z)), \end{aligned} \quad (58)$$

where C_{ij}, D_{ij} are arbitrary constants and other parameters are given by

$$\begin{aligned}
\alpha &= -\frac{i}{2}\left(k + \frac{2\omega}{x_+ - x_-}\right) & c &= 1 + 2\alpha, \\
a_{ij} &= \beta_{ij} - ik, & b_{ij} &= \beta_{ij} - \frac{2i\omega}{x_+ - x_-}, \\
\beta_{t\theta}^\pm &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4k^2\left(1 - \frac{1}{\lambda^2}\right) - 4M_{11}}, \\
\beta_{\theta\theta}^\pm &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4k^2\left(1 - \frac{1}{\lambda^2}\right) - 4M_{22}}.
\end{aligned} \tag{59}$$

We should mention that in the next section when we will impose the Dirichlet boundary condition, for finding a suitable solution, we will choose the plus sign for $h_{t\theta}$ and $h_{\theta\theta}$.

4.3 Solution for $h_{\theta t}$ and h_{tt}

As we mentioned before, for finding $h_{\theta t}$ and h_{tt} , firstly, we solve the (33) to find $h_{\theta\theta}$ and $h_{t\theta}$. Then, from the first two equations of (35) we obtain $h_{\theta t}$ and $h_{r\theta}$. After that, from the last three equations of (35) we obtain h_{tr} , $h_{r\theta}$ and especially h_{rt} in terms of $h_{\theta\theta}$, $h_{t\theta}$, $h_{\theta t}$ and h_{tt} . Finally, by inserting the h_{rt} in the third equation of (35) we obtain h_{tt} . Notice that we use the ansatz (25). The solution may be written as

$$\begin{aligned}
h_{\theta t} &= \frac{k}{A} \frac{d}{dx} h_{t\theta} + \frac{\omega}{A} \frac{d}{dx} h_{\theta\theta} - \frac{\omega B + kC}{A} h_{t\theta} - \frac{\omega D + kE}{A} h_{\theta\theta}, \\
h_{tt} &= \frac{1}{A} \frac{d}{dx} h_{\theta t} + \frac{F}{A} \frac{d}{dx} h_{\theta\theta} - \frac{G}{A} h_{\theta t} - \frac{H}{A} h_{t\theta} - \frac{I}{A} h_{\theta\theta},
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
A &= \omega \Gamma_{x\theta}^t + k \left(\Gamma_{t\theta}^x g_{xx} g^{t\theta} - m \epsilon^{tr\theta} g_{xx} g_{t\theta} \right), \\
\bar{A} &= \frac{g_{xx}}{m} \epsilon^{xt\theta} \left(-k^2 + \Gamma_{t\theta}^x \Gamma_{x\theta}^t + m^2 g_{\theta\theta} \right), \\
B &= m \epsilon^{\theta xt} g_{xx} g_{\theta\theta}, \\
C &= \Gamma_{x\theta}^\theta + g_{xx} g^{t\theta} \Gamma_{t\theta}^x, \\
D &= \Gamma_{x\theta}^\theta - m \epsilon^{\theta rt} g_{xx} g_{t\theta}, \\
E &= g_{xx} g^{\theta\theta} \Gamma_{t\theta}^x + m \epsilon^{tx\theta} g_{xx} g_{tt}, \\
F &= -\frac{\omega}{k} - \frac{\Gamma_{tt}^x}{\Gamma_{t\theta}^x},
\end{aligned}$$

$$\begin{aligned}
G &= \Gamma_{xt}^t - F\Gamma_{x\theta}^t - \frac{g_{xx}}{m}\epsilon^{xt\theta} (\omega k + \Gamma_{t\theta}^x \Gamma_{xt}^t + m^2 g_{t\theta}), \\
H &= \frac{g_{xx}}{m}\epsilon^{xt\theta} (F(k^2 - m^2 \sqrt{-g} g_{\theta\theta}) + \Gamma_{t\theta}^x \Gamma_{x\theta}^\theta), \\
I &= \frac{g_{xx}}{m}\epsilon^{xt\theta} (F(\omega k - \gamma_{x\theta}^\theta + m^2 \sqrt{-g} g_{\theta t}) + \Gamma_{t\theta}^x \Gamma_{xt}^\theta) + \Gamma_{xt}^\theta. \quad (61)
\end{aligned}$$

Using the solution (57) and the following contiguous relations between hypergeometric functions

$$\begin{aligned}
\frac{\partial}{\partial z} F_1(a, b, c; z) &= \frac{ab}{c} F_1(a+1, b+1, c+1; z), \\
azF(a+1, b+1, c+1; z) &= cF(a, b+1, c; z) - cF(a, b, c; z), \\
a(1-z)F(a+1, b, c; z) &= (c-b)F(a, b-1, c; z) - (c-a-b)F(a, b, c; z), \quad (62)
\end{aligned}$$

one can rewrite the above solutions in terms of ordinary hypergeometric functions. This calculation is straightforward but tedious. We present the result of such calculation for the case where the fields are completely symmetric in Appendix A.

4.4 Quasi Normal Modes

Quasi normal modes can be found by imposing Dirichlet boundary condition on the solutions (57) and (60).

Before that, we mention some points. Firstly, because our aim is to find the quasi-normal modes, we consider in-going waves into horizon and so we choose the constant $D_{\theta\theta} = 0$ and $D_{t\theta} = 0$. Secondly, we also choose the plus sign (57). The calculations for minus sign is similar to the case of plus sign. Thirdly, one can see that when $\lambda = 1$ then the solution (57) becomes to the solution that was found in [13]. In fact, this is due to the fact that at $\lambda = 1$ the geometry of self-dual warped AdS_3 becomes the usual BTZ geometry. So, we can use the result of computations was done in [13] for finding ratio of the coefficients $C_{\theta\theta}$ and $C_{t\theta}$. So, all coefficients of solutions of all fields can be written in terms of $C_{\theta\theta}$. Now, because of the reason which will soon be present, let us choose the $C_{\theta\theta}$ as following

$$C_{\theta\theta} = C_0(a_{\theta\theta})(a_{\theta\theta} - 1), \quad (63)$$

where C_0 is an arbitrary constant independent of $a_{\theta\theta}$ and $b_{\theta\theta}$.

After all, using the following transformation relation between hypergeometric functions

$$F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, a+b-c+1; 1-z) + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a, c-b, c-a-b+1; 1-z) \quad (64)$$

one can find the asymptotic behavior of the solution where $z \rightarrow 1$. Choosing the plus sign in (57) which means that the leading order of solutions comes from the second line of (64), one can easily find the asymptotic behavior of all terms of solution of all fields. Let us focus on one of these terms in h_{tt} solution. That is

$$h_{tt} = \frac{1}{A} \frac{d}{dx} h_{\theta t} + \dots = \frac{1}{A} \frac{d}{dx} \left(\frac{\omega}{A} \frac{d}{dx} h_{\theta\theta} \right) + \dots \quad (65)$$

Using the change of variable (54), it is not hard see that this term may be written as

$$\begin{aligned} \frac{1}{A} \frac{d}{dx} \left(\frac{\omega}{A} \frac{d}{dx} h_{\theta\theta} \right) &= C(z) z^2 \frac{d^2}{dz^2} h_{\theta\theta} + \dots \\ &\sim C(z) (a_{\theta\theta})(a_{\theta\theta} - 1) z^{\alpha+2} (1-z)^{-\beta_{\theta\theta}-2} \frac{\Gamma(c)\Gamma(a_{\theta\theta} + b_{\theta\theta} + 2 - c)}{\Gamma(a_{\theta\theta} + 1)\Gamma(b_{\theta\theta} + 1)} + \dots \end{aligned} \quad (66)$$

where $C(z)$ is a function of z and goes to a finite constant when $z \rightarrow 1$. Now, imposing Dirichlet condition on h_{tt} implies that

$$a + 1 - 2 = -n_1, \quad \text{or} \quad b + 1 = -n_2. \quad (67)$$

where n_1 and n_2 are non-negative integers. One can see that these conditions are *necessary* and *sufficient* for satisfying Dirichlet condition on all fields.

Finally, one can obtain the left and right quasi normal modes as

$$k = -i2\pi T_L l(n_1 + h_L), \quad \omega = -i2\pi T_R l(n_2 + h_R), \quad (68)$$

where

$$\begin{aligned} h_R &= +\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4k^2 \left(1 - \frac{1}{\lambda^2}\right) - 4M_{22}} \\ h_L &= -\frac{3}{2} + \frac{1}{2} \sqrt{1 - 4k^2 \left(1 - \frac{1}{\lambda^2}\right) - 4M_{22}}. \end{aligned} \quad (69)$$

Notice that from (69) one has $h_R - h_L = +2$. Note also that if one chooses $C_{\theta\theta} = C_0(b_{\theta\theta})(b_{\theta\theta} - 1)$ then will find $h_R - h_L = -2$.

The above result for conformal weight are in precise agreement with the calculations in [hidden] where the conformal weight obtained using an algebraic way from highest-weight mode.

At the end, one can also do the above calculations for the extremal warped AdS_3 black hole. The solution of equation of motion are presented in appendix B.

5 Higher Spin on Self-dual Warped AdS_3

In this section, we discuss how we can find the conformal weight and quasi normal modes of a field with arbitrary spin in self dual warped AdS_3 background. The Key point is that, as for the spin 2 case, for finding the conformal weight and quasi-normal modes of higher spin fields, it is sufficient to find the solution of $h_{\theta\theta\dots\theta}$ equation of motion. So, let us firstly find this solution in the next section.

5.1 Solution for $\Phi_{\theta\theta\dots\theta}$

The equation of motion for $\Phi_{\theta\theta\dots\theta}$ is given by (33). So, we should evaluate ∇^2 as follows⁶

$$\begin{aligned}\nabla^2\Phi_{\theta\theta\dots\theta} = \Delta\Phi_{\theta\theta\dots\theta} & - \frac{1}{\sqrt{-g}}(\partial_\alpha\sqrt{-g}g^{\alpha\beta}\Gamma_{\alpha\theta}^\sigma)\Phi_{\theta\dots\sigma\dots\theta} \\ & - 2g^{\alpha\beta}\Gamma_{\alpha\theta}^\sigma\nabla_\beta\Phi_{\theta\dots\sigma\dots\theta}, \\ & - g^{\alpha\beta}\Gamma_{\beta\theta}^\rho\Gamma_{\alpha\rho}^\sigma\Phi_{\theta\dots\sigma\dots\theta} \\ & - g^{\alpha\beta}\Gamma_{\alpha\theta}^\sigma\Gamma_{\beta\theta}^\rho\Phi_{\theta\dots\sigma\dots\rho\dots\theta}\end{aligned}\tag{70}$$

where the Δ is the scalar Laplacian (41).

Again, using the first order equation (2) and the constraint (29), we are able to rewrite the $\nabla_i\Phi_{\nu_1\dots j\dots\nu_s}$ and $\Phi_{\nu_1\dots x\dots x\dots\nu_s}$ in terms of $\Phi_{\theta\theta\nu_3\dots\nu_s}$, $\Phi_{t\theta\nu_3\dots\nu_s}$, $\Phi_{\theta t\nu_3\dots\nu_s}$ and $\Phi_{tt\nu_3\dots\nu_s}$ without any x indices.

So, using the ansatz

$$\Phi_{\theta\theta\dots\theta} = e^{-i(\omega t - k\theta)} R_{\theta\theta\dots\theta}$$

⁶In this section, by expressions similar to $\Gamma_{\mu\nu}^\sigma\Phi_{\dots\sigma\dots}$ we mean that $\Gamma_{\mu\nu}^\sigma\phi_{\sigma\nu_2\dots\nu_s} + \Gamma_{\mu\nu}^\sigma\phi_{\nu_1\sigma\dots\nu_s} + \dots + \Gamma_{\mu\nu}^\sigma\phi_{\nu_1\nu_2\dots\sigma}$.

the final differential equation for $\Phi_{\theta\theta\ldots\theta}$ reads as

$$z(1-z)\frac{\partial^2}{\partial z^2}R_{\theta\theta\ldots\theta} + (1-z)\frac{\partial}{\partial z}R_{\theta\theta\ldots\theta} + \left(Q(z) + \frac{\widetilde{M}_{22}}{1-z}\right)R_{\theta\theta\ldots\theta} = 0, \quad (71)$$

where we have used (54) and

$$\widetilde{M}_{22} = -(m - s\frac{\lambda}{2})^2 + \frac{\lambda^2}{4}. \quad (72)$$

The solution of the above equation with in-going condition on horizon is

$$R_{\theta\theta\ldots\theta} = \widetilde{C}_{\theta\theta} z^\alpha (1-z)^{\widetilde{\beta}_{\theta\theta}+1} F_1(\widetilde{\beta}_{\theta\theta} - ik + 1, \widetilde{\beta}_{\theta\theta} - \frac{2i\omega}{x_+ - x_-} + 1, c; z) \quad (73)$$

where

$$\widetilde{\beta}_{\theta\theta} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4k^2(1 - \frac{1}{\lambda^2}) - 4\widetilde{M}_{22}}, \quad (74)$$

and α, c are given in (59).

5.2 Quasi-Normal Modes

For imposing Dirichlet boundary conditions on all fields, as for spin 2 case, we focus on $\Phi_{tt\ldots t}$. The equation involving $\Phi_{tt\ldots t}$ is

$$\partial_x \Phi_{\theta t\ldots t} = \partial_\theta \Phi_{xt\ldots t} + \Gamma_{(\theta t\ldots t)} + m_{(\theta t\ldots t)}, \quad (75)$$

where

$$\begin{aligned} \Gamma_{(\theta t\ldots t)} &= \Gamma_{r\theta}^\lambda \Phi_{\theta\ldots\lambda\ldots} - \Gamma_{\theta t}^\lambda \Phi_{r\ldots\lambda\ldots}, \\ m_{(\theta t\ldots t)} &= -mg_{xx}\epsilon^{\theta rt}(g_{t\theta}\Phi_{\theta t\ldots t} - g_{\theta\theta}\Phi_{t\ldots t}). \end{aligned} \quad (76)$$

In the above expression, one can find the mostly-t field $\Phi_{tt\ldots t}$ in terms of fields with less t indices. So, using (76), among the many terms, one finds a term with maximum power of z and derivative of $\Phi_{\theta\theta\ldots\theta}$ as

$$\Phi_{tt\ldots t} \sim z^s \frac{d^s}{dz^s} \Phi_{\theta\theta\ldots\theta} + \cdots. \quad (77)$$

Now, choosing

$$C_{\theta\theta\ldots\theta} = C_0(\tilde{a}_{\theta\theta}) \cdots (\tilde{a}_{\theta\theta} - s + 1)$$

and using the asymptotic behavior of hypergeometric functions and imposing Dirichlet condition on $\Phi_{t\dots t}$, one finds the necessary and sufficient conditions in which all field being zero at infinity are

$$\tilde{a}_{\theta\theta} + s - 1 = -n, \quad \text{or} \quad \tilde{b}_{\theta\theta} + 1 = -n, \quad (78)$$

where here n is a non-negative integer.
and obtain

$$k = -i2\pi T_L l(n_1 + h_L), \quad \omega = -i2\pi T_R l(n_2 + h_R) \quad (79)$$

where

$$\begin{aligned} h_R &= +\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4k^2(1 - \frac{1}{\lambda^2}) - 4\widetilde{M}_{22}} \\ h_L &= -\frac{2s-1}{2} + \frac{1}{2} \sqrt{1 - 4k^2(1 - \frac{1}{\lambda^2}) - 4\widetilde{M}_{22}}. \end{aligned} \quad (80)$$

6 Conclusion

In this paper, we have studied the dynamics of a rank s field in a 3 dimensional self-dual warped AdS_3 background. We have firstly considered the situation in which the field is totally symmetric. We discussed that imposing the harmonic gauge constraint and tracelessness condition may not satisfy for a symmetric rank s field simultaneously. So, we have considered a general rank s field which should satisfy equation of motion (2). Although, the fields do not satisfy the symmetric and tracelessness conditions but we supposed that all modes satisfy weaker constraint (29). This constraint greatly help us to proceed in calculation. In fact, We were able to obtain the equations of motion for $h_{\theta\theta}$ and $h_{t\theta}$ modes(for a rank 2 field) which are coupled non-linear differential equations decoupled from the other modes. By a suitable redefinition of the fields, one can decouple the equations and find the exact solutions which are in terms of hypergeometric functions. Having found the solution, we have found the quasi-normal modes which are in-going waves at horizon and satisfy Dirichlet boundary condition at infinity.

At the end, we have extended our computations for a general rank s field and found the quasi-normal modes.

Due to less symmetry of the warped AdS_3 geometry, all aspects of such geometries are unknown and many open problems are yet unsolved. For example, the physics of geometry, fermions dynamics on this background (work in progress), its black holes and perturbations around the vacuum solution, the conformal field theory dual to this geometry and its dictionary are some interesting problems which should be studied.

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Appendix

A Solution for Symmetric Case

Using the solution $h_{\theta\theta} = z^\alpha(1-z)^{\beta+1}F(a+1, b+1, c; z)$ and (2) for a symmetric field $h_{\mu\nu}$ and (62), one may obtain the solution for other components

$$\begin{aligned} h_{\theta t} &= A_1 z \frac{dh_{\theta\theta}}{dz} + \frac{A_2 + A_3 z}{1-z} h_{\theta\theta} \\ &= z^\alpha(1-z)^\beta \frac{1}{A} \left(\sum_{i,j=0}^1 C_{\theta t}^{ij} F(a+1-i, b+1-j, c; z) \right), \end{aligned} \quad (81)$$

$$\begin{aligned} h_{tt} &= B_1 z \frac{dh_{\theta t}}{dz} + \frac{B_2 + B_3 z}{1-z} h_{\theta t} + \frac{B_4 + B_5 z + B_6 z^2}{(1-z)^2} h_{\theta\theta} \\ &= z^\alpha(1-z)^{\beta-1} \frac{1}{A\bar{A}} \left(\sum_{i,j=0}^2 C_{tt}^{ij} F(a+1-i, b+1-j, c; z) \right), \end{aligned} \quad (82)$$

where

$$\begin{aligned} C_{\theta t}^{00} &= (3\beta + 2 + W), \\ C_{\theta t}^{01} &= (a - 2\beta), \\ C_{\theta t}^{10} &= -\frac{(a - 2\beta)}{(2\beta)} C_{\theta t}^{11} = -\frac{(2\beta)}{(a)} (\alpha - \beta - a - 2 - B), \end{aligned}$$

$$\begin{aligned}
C_{tt}^{00} &= \left(C_{t\theta}^{00}(3\beta + 1 + \overline{W} - 2(1 - \lambda^2)\tilde{A}) + 2(1 - \lambda^2)\bar{x}_0 A \right), \\
C_{tt}^{01} &= 2(a - 2\beta) \left(3\beta + 1 - \lambda m - (1 - \lambda^2)\tilde{A} \right), \\
C_{tt}^{10} &= \frac{(2\beta)}{(a)} \left(\frac{a}{2\beta} C_{t\theta}^{10}(3\beta + \overline{W} - 2(1 - \lambda^2)\tilde{A}) - 2(1 - \lambda^2)\bar{x}_0 A \right. \\
&\quad \left. - C_{t\theta}^{00}(-\alpha + \beta + a + 1 + \overline{B} - (1 - \lambda^2)\tilde{A}) \right), \\
C_{tt}^{02} &= (a - 2\beta + 1)(a - 2\beta), \\
C_{tt}^{20} &= -\frac{(2\beta)(1 - 2\beta)}{(a)(a - 1)} \left(\frac{a}{2\beta} C_{\theta t}^{10}(\alpha - \beta - a - \overline{B} + (1 - \lambda^2)\tilde{A}) + (1 - \lambda^2)A\tilde{B} \right) \\
C_{tt}^{11} &= \frac{(a - 2\beta)}{(a)} \left(\frac{a}{2\beta} C_{\theta t}^{10}(5\beta - 1 + \overline{B} - 2(1 - \lambda^2)\tilde{A}) - 2(1 - \lambda^2)A\tilde{B} \right. \\
&\quad \left. + (-5\beta + 1)(-\alpha + \beta + a + 1 + \overline{B} - (1 - \lambda^2)\tilde{A}) \right) \\
C_{tt}^{12} &= \frac{(a - 2\beta + 1)(a - 2\beta)}{(a)} \left(2\alpha - 2\beta - 2a - 3 - B - \overline{B} + (1 - \lambda^2)\tilde{A} \right) \\
C_{tt}^{21} &= -\frac{(1 - 2\beta)}{(a + 1 - 2\beta)} C_{tt}^{22} = -\frac{(a - 2\beta)}{(2\beta)} C_{tt}^{20}. \tag{83}
\end{aligned}$$

Here, we have defined

$$\begin{aligned}
W &= -\lambda m - \frac{\lambda^2}{2}, & \overline{W} &= -\lambda m + \frac{\lambda^2}{2}, \\
A &= \frac{W^2 + k^2}{2W\bar{x}_0}, & \overline{A} &= \frac{\overline{W}^2 + k^2}{2\overline{W}\bar{x}_0}, \\
B &= \frac{W^2\bar{x}_0 + k\omega}{2W\bar{x}_0}, & \overline{B} &= \frac{\overline{W}^2\bar{x}_0 + k\omega}{2\overline{W}\bar{x}_0}, \\
\tilde{A} &= \frac{W\overline{W} - k^2}{2W\overline{W}}, & \tilde{B} &= \frac{W\overline{W}\bar{x}_0 - k\omega}{2W\overline{W}}. \tag{84}
\end{aligned}$$

and we have used the relations (59).

B Extremal Case

In this appendix, we consider the extremal self-dual warped AdS_3 black hole. Defining $x_+ = x_- = x_0$, the equations of motion of $\mathcal{H}_{t\theta}$ and $\mathcal{H}_{\theta\theta}$ read as

$$\begin{aligned}
(x - x_0)^2 \frac{\partial^2}{\partial x^2} \mathcal{H}_{t\theta} + 4(x - x_0) \frac{\partial}{\partial x} \mathcal{H}_{t\theta} + (Q(x) + 2 + M_{11}) \mathcal{H}_{t\theta} &= 0, \\
(x - x_0)^2 \frac{\partial^2}{\partial x^2} \mathcal{H}_{\theta\theta} + 2(x - x_0) \frac{\partial}{\partial x} \mathcal{H}_{\theta\theta} + (Q(x) + M_{22}) \mathcal{H}_{\theta\theta} &= 0. \tag{85}
\end{aligned}$$

where $Q(x)$ given by (46). The above equations can be written in the form of Whittaker differential equation. Then, the solutions can be written in terms of Whittaker functions as following

$$\begin{aligned}\mathcal{H}_{t\theta}(x) &= \left(\tilde{C}_{t\theta} WhittakerW(-ik, \frac{1}{2} + \beta_{ij}, \frac{2i\omega}{x - x_0}) \right. \\ &\quad \left. + \tilde{D}_{t\theta} WhittakerM(-ik, \frac{1}{2} + \beta_{ij}, \frac{2i\omega}{x - x_0}) \right) \left(\frac{1}{x - x_0} \right),\end{aligned}\quad (86)$$

$$\begin{aligned}\mathcal{H}_{\theta\theta}(x) &= \left(\tilde{C}_{\theta\theta} WhittakerW(-ik, \frac{1}{2} + \beta_{ij}, \frac{2i\omega}{x - x_0}) \right. \\ &\quad \left. + \tilde{D}_{\theta\theta} WhittakerM(-ik, \frac{1}{2} + \beta_{ij}, \frac{2i\omega}{x - x_0}) \right),\end{aligned}\quad (87)$$

where

$$\begin{aligned}WhittakerM(\mu, \nu; z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\nu} hypergeom(\frac{1}{2} + \nu - \mu, 1 + 2\nu; z), \\ WhittakerW(\mu, \nu; z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\nu} KummerU(\frac{1}{2} + \nu - \mu, 1 + 2\nu; z).\end{aligned}$$

Having found the $\mathcal{H}_{t\theta}$ and $\mathcal{H}_{\theta\theta}$ and using the solutions (60) for the case $x_+ = x_- = x_0$, one can find the solution for all other modes.

C Generalized Contiguous Relations

Using (62), we obtains the generalized version of recursion relations (62) as

$$\begin{aligned}\frac{\partial^n}{\partial z^n} F(a, b, c; z) &= \frac{(a)_n (b)_n}{(c)_n} F(a + n, b + n, c + n; z), \\ \frac{(a)_n}{(c)_n} z^n F(a + n, b + n, c + n; z) &= \\ &= \sum_{m=0}^n \binom{n}{m} (-1)^m F(a, b + n - m, c; z) \\ (1 - z)^n F(a + 1, b + 1, c; z) &= \\ \sum_{m=0}^n \prod_{i=0}^{n-m} \prod_{j=0}^m (-1)^m \binom{n}{m} \frac{\Gamma(a - n)}{\Gamma(a)} (c - b + i - 1)(c - b - a - j + 1) \\ &\quad \times F_1(a - n + 1, b - n + m + 1, c; z).\end{aligned}\quad (88)$$

where $(a)_n = (a)(a + 1) \cdots (a + n - 1)$ and $(a)_0 = 1$.

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